Spinor Representations of $U_q(\hat{\mathfrak{gl}}(n))$ and Quantum Boson-Fermion Correspondence

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abstract

This is an extension of quantum spinor construction in [DF2]. We define quantum affine Clifford algebras based on the tensor category and the solutions of q-KZ equations, construct quantum spinor representations of $U_q(\hat{\mathfrak{gl}}(n))$ and explain classical and quantum boson-fermion correspondence.

I. Introduction.

The independent discovery of a q-deformation of universal enveloping algebra of an arbitrary Kac-Moody algebra by Drinfeld [D1] and Jimbo [J1] immediately raised numerous questions about q-deformations of various structures associated to Kac-Moody algebras. A major step in this direction was the work by Lusztig [L], who obtained a q-deformation of the category of highest weight representations of Kac-Moody algebras for generic or formal parameter q. There were a number of successful results on q-deformation of various mathematical structures of finite dimensional and affine Lie algebras [D2], [J2], [FJ], [H], [FR], etc. However each particular problem required its own special insight and in some cases presented formidable difficulties.

In [DF2], we propose an invariant approach, which was also stressed in [FRT], where a q-analogue of matrix realization of classical Lie algebras was given. We manage to use such an approach to define quantum Clifford and Weyl algebras using general representation theory of quantum groups. We show that the explicit formulas for quantum Clifford and Weyl algebras match the ones actively studied in physics literature (see e.g. [WZ], [K]). Using those quantum algebras, we construct spinor and oscillator representations of quantum groups of classical types and recover all the relevant formulas for the quantum construction. Uniqueness arguments from representation theory allow us to justify that the quadratic expressions in quantum Clifford and Weyl algebras provide the desired representations. An explicit verification of Serre's relations for quantum groups lead to rather involved formulas (cf. [H]). The key idea consists of reformulating familiar classical constructions entirely in terms of the tensor category of highest weight representations and then, using Lusztig's result on q-deformation of this category to define the corresponding quantum structures. In the quantum case, we often need a quasitriangular structure of the tensor category introduced by Drinfeld [D3], since it plays the role of the symmetric structure in the classical case. In particular, we would like to emphasize the central role of the universal Casimir operator implied by the quantum structure.

The motivation to develop such an invariant approach for the quantum groups corresponding to the simple finite dimensional algebras is surely to apply it to general cases, especially affine quantum groups. In this paper, we parallelly extend the idea to the cases of the spinor representations of quantum affine groups $U_q(\hat{\mathfrak{gl}}(n))$

present the corresponding results of the cases of $\hat{\mathfrak{o}}(n)$ and $\hat{\mathfrak{s}}p(2n)$ as for the cases of $\mathfrak{o}(n)$ and $\mathfrak{s}p(2n)$ in [DF2].

For undeformed affine Lie algebras, Garland showed [G] that the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ admits a natural realization as a central extension of the corresponding loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

For classical affine Lie algebras, we have the Frenkel-Kac construction to obtain so-called bosonic representations by vertex operators in terms of Heisenberg algebra. There also exists another family of representations, the fermionic representations [F] [FF] in terms of affine Clifford algebras. For these two families of representations, there exists the well known boson-fermion correspondence [F1] to show their equivalent relations.

As in the undeformed classical cases, with the canonical basis for the n-dimension representation of $\mathfrak{g}l(n)$, starting from abstract representations without any concrete realization, we can write down formulas that completely parallel to the ones in [DF2] to obtain the isomorphism between the algebra generated by intertwiners and affine Clifford algebra, and derive the spinor construction. This provides a structural explanation of the boson-fermion correspondence.

For the quantum case, Faddeev, Reshetikhin and Takhtajan [FRT2] have shown how to extend their realization of $U_q(\mathfrak{g})$ to the quantum loop algebra $U_q(\mathfrak{g} \otimes [t, t^{-1}])$ via a canonical solution of the Yang-Baxter equation depending on a parameter $z \in \mathbb{C}$. The first realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and its special degeneration called the Yangian were obtained by Drinfeld [D2].

Later Reshetikhin and Semenov-Tian-Shansky [RS] incorporated the central extension in the previous construction of [FRT2] to obtain the second realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$.

As for the cases of deformation of finite dimensional Lie algebras, with the theory of Lusztig of the q-deformation of the category of highest weight representations, we construct the spinor representation of $U_q(\hat{\mathfrak{g}}l(n))$ with intertwiners completely parallel to the case of the quantum groups of classical types as explained in [DF2].

On the other hand, with the Heisenberg algebra, Frenkel and Jing [FJ] constructed representations of $U_q(\hat{\mathfrak{s}}l(n))$ by vertex operators in terms of Drinfeld's realization, which could be extended to $U_q(\hat{\mathfrak{g}}l(n))$. This gives us the quantized bosonic construction.

In [DF], we explicitly established a relation between two realizations of the quantum affine algebra $U_q(\hat{\mathfrak{g}\mathfrak{l}}(n))$. We show that the realization of Drinfeld's construction can be naturally established in the Gauss decomposition of a matrix composed of elements of the quantum affine algebra. With this isomorphism, we are able to establish the quantum boson-fermion correspondence for the case of $U_q(\hat{\mathfrak{g}}l(n))$ by the concrete isomorphism between those two representations via the intertwiners, whose bosonization is partially solved in [Ko].

Hayashi constructed fermionic representations of $U_q(\hat{\mathfrak{s}}l(n))$, where however he only obtained the realization of the basic generators. From the point view of this paper, his definition of the quantized fermions does not reveal the new structure quantum fermions possess.

The quantum Clifford algebras we define are closely related to massive quantum field theory[Sm]. We will construct a realization of an algebra, which resmebles the algebra to define form factors in massive quantum field theory based on the

This paper is arranged as follows. We present the basic idea of this paper in Section 1. Then in Section 2, we will present different realizations of $U_q(\hat{\mathfrak{gl}}(n))$. In section 3, we will reconstruct classical spinor representations and boson-fermion correspondence. Section 4 will give the construction of spinor representations of $U_q(\hat{\mathfrak{gl}}(n))$ and explain the quantum boson-fermion correspondence. We will also discuss the connection with the theory of form factors in massive quantum field theory.

1. Univeral Casimir Operators of Affine Quantum Groups

The definitions of quantum groups corresponding to the affine Lie algebras, which were given by Drinfeld [D1] and Jimbo [J1] in terms of generators and quantized relations corresponding to the affine Cartan matrices, are very simple.

Definition 1.1. *Let (a_{ij}) be a Cartan matrix for an affine Lie algebra $\hat{\mathfrak{g}}$ [Ka]. Let $(d_0, \dots, d_n)$ be a vector with integer entries $d_i \in \{1, 2, 3, 4\}$ such that $(d_i a_{ij})$ is symmetric. Let q be an indeterminate. $U_q(\hat{\mathfrak{g}})$ is an associative algebra on $\mathbb{C}[q, q^{-1}]$ with generators E_i, F_i, K_i, K_i^{-1} $(i = 0, 1.., n)$ and the relations are:*

$$K_i K_j = K_j K_i \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$
$$K_i E_j = q^{d_i a_{ij}} E_j K_i, \qquad K_i F_j = q^{-d_i a_{ij}} F_j K_i$$
$$E_i E_j - F_j E_i = \delta_{ij} \frac{K_i - K_j^{-1}}{q^{d_i} - q^{-d_i}}$$

(1.1)
$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0, \qquad if \ i \neq j,$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0, \qquad if \ i \neq j,$$

where for integers $N, M, d \geq 0$, we define

(1.2)
$$[N]_d! = \prod_{a=1}^{N} \frac{q^{da} - q^{-da}}{q^d - q^{-d}}, \qquad \begin{bmatrix} M + N \\ N \end{bmatrix}_d = \frac{[M + N]_d!}{[M]_d![N]_d!}.$$

The quantum group $U_q(\hat{\mathfrak{g}})$ has a noncocommutative Hopf algebra structure with comultiplication Δ, antipode S and counit ε defined by

(1.3)
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(K_i) = K_i \otimes K_i,$$
$$S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i, \qquad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1.$$

We define an automorphism D_z of $U_q(\hat{\mathfrak{g}})$ as

and D_z fixes all other generators. We also define the map $\Delta_z(a) = (D_z \otimes id)\Delta(a)$ and $\Delta'_z(a) = (D_z \otimes id)\Delta'(a)$, where $a \in U_q(\hat{\mathfrak{g}})$ and Δ' denotes the opposite comultiplication.

Let d be an operator such that d commutes with all other elements but have the relation that

$$[d, E_0] = E_0, [d, F_0] = -F_0.$$

It is clear that action of D_z is equivalent to conjugation by z^d.

For the algebra generated by $U_q(\hat{\mathfrak{g}})$ and the operator d, which we denote by $U_q(\tilde{\mathfrak{g}})$, from the theory of Drinfeld[D2], we know that it has a universal R-matrix $\bar{\mathfrak{R}}$.

Proposition 1.1. [D2] *There exists an element $\bar{\mathfrak{R}}$ in $U_q^+(\tilde{\mathfrak{g}})\hat{\otimes}U_q^-(\tilde{\mathfrak{g}})$ such that $\bar{\mathfrak{R}}$ satisfies the properties:*

(1.4)
$$\bar{\mathfrak{R}}\Delta(a) = \Delta^{\mathrm{op}}(a)\bar{\mathfrak{R}},$$
$$(\Delta \otimes id)(\bar{\mathfrak{R}}) = \bar{\mathfrak{R}}_{13}\bar{\mathfrak{R}}_{23},$$
$$(id \otimes \Delta)(\bar{\mathfrak{R}}) = \bar{\mathfrak{R}}_{13}\bar{\mathfrak{R}}_{12},$$

where $a \in U_q(\mathfrak{g})$, Δ^{op} denotes the opposite comultiplication, $\bar{\mathfrak{R}}_{12} = \sum_i a_i \otimes b_i \otimes 1 = \bar{\mathfrak{R}} \otimes 1, \bar{\mathfrak{R}}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \bar{\mathfrak{R}}_{23} = \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes \bar{\mathfrak{R}}.$

Here $U_q^+(\tilde{\mathfrak{g}})$ is the subalgebra generated by E_i, K_i and d and $U_q^-(\tilde{\mathfrak{g}})$ is the subalgebra generated by F_i, K_i and d.

Let

$$\bar{\mathfrak{R}}(z) = (D_z \otimes \mathrm{id})\bar{\mathfrak{R}}.$$

Let C be the central element corresponding to the central extension of the quantum affine algebra. Let $\mathfrak{R}(z) = q^{-d\otimes C - C \otimes d}\bar{\mathfrak{R}}(z)$, then we have

Corollary 1.1. $\mathfrak{R}(z) \in U_q(\hat{\mathfrak{b}^+})\hat{\otimes}U_q(\hat{\mathfrak{b}^-}) \otimes \mathbb{C}[[z]]$, *such that*

$$\mathfrak{R}(z)\Delta_z(a) = (D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\Delta'_z(a)\mathfrak{R}(z),$$

$$(\Delta \otimes I)\mathfrak{R}(z) = \mathfrak{R}_{13}(zq^{C_2})\mathfrak{R}_{23}(z),$$

(1.5)
$$(I \otimes \Delta)\mathfrak{R}(z) = \mathfrak{R}_{13}(zq^{-C_2})\mathfrak{R}_{12}(z),$$

$$\mathfrak{R}_{12}(z)\mathfrak{R}_{13}(zq^{C_2}/w)\mathfrak{R}_{23}(w) = \mathfrak{R}_{23}(w)\mathfrak{R}_{13}(zq^{-C_2}/w)\mathfrak{R}_{12}(z).$$

Here $C_1 = C \otimes 1$, $C_2 = 1 \otimes C$, $U_q(\hat{\mathfrak{b}^+})$ *is the subalgebra generted by E_i, K_i and $U_q(\mathfrak{b}^-)$ is the subalgebra generrated by F_i, K_i.*

Let

$$\mathfrak{R} = (D_z^{-1} \otimes 1)\mathfrak{R}(z).$$

Corollary 1.2. *Let* $\mathfrak{C} = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\mathfrak{R}_{21})\mathfrak{R}$. *Then*

$$(1.6) \qquad\qquad \mathfrak{C}\Delta(a) = ((D_{q^{C_2}}^{-2} \otimes D_{q^{C_1}}^{-2})\Delta(a))\mathfrak{C}.$$

Proof. From the property of $\mathfrak{R}(z)$, we know that

$$(1.7) \qquad\qquad \mathfrak{R}\Delta(a) = (D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\Delta'(a)\mathfrak{R}$$

Thus we have

$$(1.8) \qquad\qquad \mathfrak{R}_{21}\Delta'(a) = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\Delta(a))\mathfrak{R}_{21},$$

and

$$(1.9) \quad ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}))(\mathfrak{R}_{21}\Delta'(a)) = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}))(((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\Delta(a))\mathfrak{R}_{21}).$$

Thus we get that

$$(1.10)$$
$$(D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}\mathfrak{R}_{21})(D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}\Delta'(a)) = (D_{q^{C_2}}^{-2} \otimes D_{q^{C_1}}^{-2}\Delta(a))(D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}\mathfrak{R}_{21}).$$

Thus we obtain the proof.

We note that $(q^{C_2} \otimes q^{C_1})$ is invariant under the permutation.

This proposition shows that the action of $\mathfrak{C} = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\mathfrak{R}_{21})\mathfrak{R}$ on a tensor product of two modules is an intertwiner which, however, shifts the actions of E_0 and F_0 by the constants $q^{\mp 2C_2} \otimes q^{\mp 2C_1}$ respectively.

We should also notice that $D_z \otimes D_z$ acts invariantly on $\mathfrak{R}$. Let $\mathfrak{R}_{21}(z) = (D_z \otimes 1)\mathfrak{R}_{21}$. Note that $\mathfrak{R}_{21}(z)$ is not equal to $P(\mathfrak{R}(z))$, where P is the permutation operator.

Let V be a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$.

Let

$$(1.11a) \qquad\qquad \bar{L}^+(z) = (\mathrm{id} \otimes \pi_V)(\mathfrak{R}_{21}(z)),$$

$$\bar{L}^{-^{-1}}(z) = (\mathrm{id} \otimes \pi_V)\mathfrak{R}(z^{-1}),$$

$$\mathfrak{L}^+(z) = (\pi_V \otimes \mathrm{id})(\mathfrak{R}^{-1}(z)),$$

$$\mathfrak{L}^{-^{-1}}(z) = (\pi_V \otimes \mathrm{id})\mathfrak{R}_{21}^{-1}(z^{-1}).$$

We have that

$$(1.11b) \qquad \bar{L}^+(z)P\mathfrak{L}^+(z^{-1}) = 1, \quad \bar{L}^{-^{-1}}(z)P\mathfrak{L}^{-^{-1}}(z^{-1}) = 1,$$

where P is the permutation operator.

$U_q(\hat{\mathfrak{g}})$ as an algebra is generated by operator entries of $\bar{L}^+(z)$ and $\bar{L}^{-^{-1}}(z)$, and it is also generated by operator entries of $\mathfrak{L}^+(z)$ and $\mathfrak{L}^{-^{-1}}(z)$. $\bar{L}^\pm(z)$ are used in [FR] to obtained q-KZ equation.

Let

$$(1.12a) \qquad \bar{L}(z) = (\mathrm{id} \otimes \pi_V)((1 \otimes D_{q^C}^{-1})\mathfrak{R}_{21}(z))\mathfrak{R}(z^{-1}),$$

$$(D_z \otimes 1)\bar{L}(z) = \bar{L} = (\mathrm{id} \otimes \pi_V)((1 \otimes D_{q^C}^{-1})\mathfrak{R}_{21})\mathfrak{R},$$

and

$$(1.12b) \qquad \mathfrak{L}(z) = (\pi_V \otimes \mathrm{id})((D_{q^C} \otimes 1)\mathfrak{R}^{-1}(z))\mathfrak{R}_{21}^{-1}(z^{-1}),$$

$$(1 \otimes D_z)\mathfrak{L}(z) = \mathfrak{L} = (\pi_V \otimes \mathrm{id})((1 \otimes D_{q^C})\mathfrak{R}^{-1})\mathfrak{R}_{21}^{-1}.$$

Proposition 1.2.

$$\bar{R}(\frac{z}{w})\bar{L}_1^{\pm}(z)\bar{L}_2^{\pm}(w) = \bar{L}_2^{\pm}(w)\bar{L}_1^{\pm}(z)\bar{R}(\frac{z}{w}),$$

$$\bar{R}(\frac{zq^{-C}}{w})\bar{L}_1^{+}(z)\bar{L}_2^{-}(w) = \bar{L}_2^{-}(w)\bar{L}_1^{+}(z)\bar{R}(\frac{zq^{C}}{w}),$$

(1.13a) $\quad \bar{R}(z/w)\bar{L}_1(z)\bar{R}(zq^{2C}/w)^{-1}\bar{L}_2(w) = \bar{L}_2(w)\bar{R}(zq^{-2C}/w)\bar{L}_1(z)\bar{R}(z/w)^{-1},$

$$\bar{L}(z)(id \otimes \pi_V)\Delta(a) = ((1 \otimes D_{q^C}^{-2})\Delta(a))\bar{L}(z),$$

and

(1.13b)

$$\bar{R}(\frac{z}{w})\mathfrak{L}_1^{\pm}(z)^{-1}\mathfrak{L}_2^{\pm}(w)^{-1} = \mathfrak{L}_2^{\pm}(w)^{-1}\mathfrak{L}_1^{\pm}(z)^{-1}\bar{R}(\frac{z}{w}),$$

$$\bar{R}(\frac{zq^{-C}}{w})\mathfrak{L}_1^{+}(z)^{-1}\mathfrak{L}_2^{-}(w)^{-1} = \mathfrak{L}_2^{-}(w)^{-1}\mathfrak{L}_1^{+}(z)^{-1}\bar{R}(\frac{zq^{C}}{w}),$$

$$\mathfrak{L}_1(z)\bar{R}(zq^{-2C}/w)\mathfrak{L}_2(w)\bar{R}(z/w)^{-1} = \bar{R}(z/w)\mathfrak{L}_2(w)\bar{R}(zq^{2C}/w)^{-1}\mathfrak{L}_1(z),$$

$$\mathfrak{L}(z)(id \otimes \pi_V)\Delta(a) = (D_{q^{2C}} \otimes 1\Delta(a))\mathfrak{L}(z),$$

Here $\bar{R}(z/w)$ is the image of $\mathfrak{R}(z/w)$ on $V \otimes V$.

We name $\bar{L}(z)$ and $\mathfrak{L}(z)$ universal Casimir operator and inverse universal Casimir operator of the quantum algebra respectively.

The construction of a representation $U_q(\hat{\mathfrak{g}})$ is equivalent to finding a specific realization of the operator $\bar{L}(z)$ or $\mathfrak{L}(z)$, which plays the same role as L in the case of $U_q(\mathfrak{g})$ in [DF2]. Naturally we would like to find a way to build this $\bar{L}(z)$ or $\mathfrak{L}(z)$ out of the intertwiners, just as in the case of spinor and oscillator representations of the quantum groups of types A, B, C and D in [DF2].

Let $V_{\lambda,k}$ and $V_{\lambda_1,k}$ be two highest weight representations of $U_q(\hat{\mathfrak{g}})$ with highest weight λ and λ_1 and the center C acting as a multiplication by a number k.

Let Ψ be an intertwiner:

$$\Psi : V_{\lambda_1,k} \longrightarrow V_{\lambda,k} \otimes V,$$

(1.14) $\qquad \Psi(x) = \Psi_1(x) \otimes e_1 + ... + \Psi_n(x) \otimes e_n,$

where $x \in V_{\lambda_1,k}$ and e_i is a basis for V. Let Ψ^* be an intertwiner:

$$\Psi^* : V_{\lambda,k} \longrightarrow V_{\lambda_1,k} \otimes V^*,$$

(1.15) $\qquad \Psi^*(x) = \Psi_1^*(x) \otimes e_1^* + ... + \Psi_n^*(x) \otimes e_n^*,$

where V^* is the left dual representation of V of $\hat{U}_q(\mathfrak{g})$, $x \in V_{\lambda,k}$ and e_i^* is a basis for V^*. Let us identify $V \otimes V^*$ with $\text{End}(V)$.

$(\Psi \otimes 1)\Psi^* = \Sigma \Psi_i \Psi_j^* \otimes e_i \otimes e_j^*$ gives a map

Let $\tilde{L} \in \text{End}(V_{\lambda,k}) \otimes \text{End}(V)$:

(1.16)
$$\tilde{L} = (\tilde{L}_{ij}) = ((D_{q^{-2k}}\Psi_i)\Psi_j^*),$$

where $(D_{q^{-2k}}\Psi_i)$ means shifting the evaluation representation by the constant q^{-2k}. Here we need the assumption that $\tilde{L}$ is well defined, which we define as follows:

On $V_{\lambda,k}$ and $V_{\lambda_1,k}$, we define a grading by defining the action of E_0 lifting the degree of an element by 1 and F_0 lowering the degree of an element by 1. Then Ψ_i can be written as $\Sigma_{\mathbb{Z}}\Psi_i(n)$, where $\Psi_i(n)$ shifts the degree of an element by n. Let $\Psi_i^{a,b} = \Sigma_a^b \Psi_i(n)$, $a < b$. By $\tilde{L}$ is well defined, we mean that any homogeneous component of the image of an element in $V_{\lambda,k}$ under the composite action of Ψ_j^* and $\Psi_i^{a,b}$ converges when a goes to negative infinity and b goes to positive infinity.

Proposition 1.3.

(1.17)
$$\tilde{L}\Delta(a) = ((1 \otimes D_{q^{-2k}})\Delta(a))\tilde{L},$$

where a is an element in $U_q(\hat{\mathfrak{g}})$.

Let Φ be an intertwiner:

$$\Phi : V_{\lambda_1,k} \longrightarrow V \otimes V_{\lambda,k},$$

(1.18)
$$\Phi(x) = e_1 \otimes \Phi_1(x) + ... + e_n \otimes \Phi_n(x),$$

where $x \in V_{\lambda_1,k}$ and $\{e_i\}$ is the basis for V. Let Φ^* be an intertwiner:

$$\Phi^* : V_{\lambda,k} \longrightarrow^* V \otimes V_{\lambda_1,k},$$

(1.19)
$$\Phi^*(x) = e_1^* \otimes \Phi_1^*(x) + ... + e_n^* \otimes \Phi_n^*(x),$$

where $^*V$ is the right dual representation of V of $\hat{U}_q(\mathfrak{g})$, $x \in V_{\lambda,k}$ and e_i^* is the basis for $^*V$.

By the right dual representation of V of $\hat{U}_q(\mathfrak{g})$, we mean the action of $\hat{U_q}(\mathfrak{g})$ on the dual space given by $< av', v >=< v', S^{-1}(a)v >$, for $a \in \hat{U}_q(\mathfrak{g})$, $v \in V$ and $v' \in {}^*V$.

$(1 \otimes \Phi)\Phi^* = \Sigma\Phi_i\Phi_j^* \otimes e_j^* \otimes e_i$ gives a map

$$\Phi : V_{\lambda,k} \longrightarrow^* V \otimes V \otimes V_{\lambda,k}.$$

Let us identify $^*V \otimes V$ with $\text{End}(V)$ by the map which maps the first two components of $V \otimes^* V \otimes V$ to $\mathbb{C}$ and fix the last component.

Let $\tilde{\mathfrak{L}} \in \text{End}(V) \otimes \text{End}(V_{\lambda,k})$:

(1.20)
$$\tilde{\mathfrak{L}} = (\tilde{\mathfrak{L}}_{ij}) = ((D_{q^{2k}}\Phi_j)\Phi_i^*),$$

where $(D_{q^{2k}}\Phi_i)$ means shifting the evaluation representation by constant q^{2k} and

Proposition 1.5.

(1.21) $$\tilde{\mathfrak{L}}\Delta(a) = ((D_{q^{2k}} \otimes 1)\Delta(a))\tilde{\mathfrak{L}}$$

The key idea is to identify $\tilde{L}$ with L or $\tilde{\mathfrak{L}}$ with $\mathfrak{L}$ to obtain representations out of intertwiners.

The intertwiners for the affine quantum groups are extensively studied by Kyoto school. They connected the XXZ model in statistical mechanics with the structures of the representation of quantum affine algebras via the intertwiners. Meanwhile Jimbo, Miki, Miwa and Nakayashiki [JMMN] and Koyama [K] worked on the bosonization of the intertwiners of quantum affine algebras. These results, in some sense, is trying to obtain the fermions out of bosons.

The results in [DF] enable us to obtain the explicit quantum boson-fermion correspondence via Gauss decomposition of $L^{\pm}(z)$. Our construction also brings a different but conceptional understanding to the classical boson-fermion correspondence.

On the other hand, Miki[M], Foda, Iohara, Jimbo, Kedem and Yan [FIJKMY] presented another idea to construct realizations of $\bar{L}^{\pm}$ by composition of Ψ and Φ^*.

In a subsequent paper, we will use the idea in this paper to define the corresponding quantum Clifford algebras, quantum Weyl algebras and construct spinor and oscillator representations of $U_q(\hat{\mathfrak{o}}(2n))$ and $U_q(\hat{\mathfrak{s}}p(2n))$. For the spinor representation of $U_q(\hat{\mathfrak{o}}(2n))$, only the bosonic realization [B] is available. We expect to build the boson-fermion correspondence in a similar but simpler way.

2. Realizations of Affine Quantum Groups.

The Drinfeld-Jimbo definition of quantum groups by generators and relations is valid for an arbitrary generalized Cartan matrix. In particular, the choice of extended Cartan matrix of type $A_n^{(1)}$ yields the quantum affine algebra $U_q(\widehat{\mathfrak{sl}(n)})$. Drinfeld found in [D4] another realization of the quantum affine algebras, which to a certain degree plays the role of loop algebra realization in the undeformed case. We extend Drindeld's construction to the quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$.

Definition 2.1. $U_q(\widehat{\mathfrak{gl}(n)})$ *is an associative algebra with unit 1 and generators*

(2.1) $\{X_{ik}^{\pm}, k_{jl}^+, k_{jm}^-, q^{\pm\frac{1}{2}c} \mid i = 1, \ldots, n-1, j = 1, \ldots, n, k \in \mathbf{Z}, l \in \mathbf{Z}_+, m \in -\mathbf{Z}_+\}$

satisfying relations in terms of the following generating functions in a formal variable z:

(2.2)
$$X_i^{\pm}(z) = \sum_{k \in \mathbf{Z}} X_{ik}^{\pm} z^{-k}$$
$$k_j^+(z) = \sum_{l \in \mathbf{Z}_+} k_{jl}^+ z^{-l}$$
$$k_j^-(z) = \sum_{m \in -Z_+} k_{jm}^- z^{-m}.$$

The generators $q^{\pm\frac{1}{2}c}$ are central and mutually inverse. The other relations are:

(2.3)

$$k_i^{\pm}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\pm}(z)$$
$$k_i^{+}(z)k_i^{-}(w) = k_i^{-}(w)k_i^{+}(z).$$

$$\frac{z_{\mp} - w_{\pm}}{z_{\mp}q^{-1} - w_{\pm}q}k_i^{\mp}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\mp}(z)\frac{z_{\pm} - w_{\mp}}{z_{\pm}q^{-1} - w_{\mp}q}, \quad if \quad j > i.$$

$$\begin{cases} k_i^{\pm}(w)^{-1}X_j^{\pm}(z)k_i^{\pm}(z) = X_j^{\pm}(z) & if \quad i-j \le -1 \quad or \\ k_i^{\pm}(w)^{-1}X_j^{\mp}(z)k_i^{\pm}(w) = X_j^{\mp}(z) & i-j \ge 2 \end{cases}.$$

$$k_i^{\pm}(z)^{-1}X_i^{-}(w)k_i^{\pm}(z) = \frac{z_{\mp}q^{-1} - wq}{z_{\mp} - w}X_i^{-}(w)$$

$$k_{i+1}^{\pm}(z)^{-1}X_i^{-}(w)k_{i+1}^{\pm}(z) = \frac{z_{\mp}q - wq^{-1}}{z_{\mp} - w}X_i^{-}(w)$$

$$k_i^{\pm}(z)X_i^{+}(w)k_i^{\pm}(z)^{-1} = \frac{z_{\pm}q^{-1} - wq}{z_{\pm} - w}X_i^{+}(w)$$

$$k_{i+1}^{\pm}(z)X_i^{+}(w)k_{i+1}^{\pm}(z)^{-1} = \frac{z_{\pm}q - wq^{-1}}{z_{\pm} - w}X_i^{+}(w).$$

(2.4)
$$(zq - wq^{-1})X_i^{-}(z)X_i^{-}(w) = X_i^{-}(w)X_i^{-}(z)(zq^{-1} - wq)$$
$$(zq^{-1} - wq)X_i^{+}(z)X_i^{+}(w) = X_i^{+}(w)X_i^{+}(z)(zq - wq^{-1}).$$

$$(z - w)X_i^{+}(z)X_{i+1}^{+}(w) = (zq^{-1} - wq)X_{i+1}^{+}(w)X_i^{+}(z)$$
$$(zq^{-1} - wq)X_i^{-}(z)X_{i+1}^{-}(w) = (z - w)X_{i+1}^{-}(w)X_i^{-}(z) \qquad .$$
$$[X_i^{\pm}(z), X_j^{\pm}(w)] = 0, \qquad for \quad A_{ij} = 0$$

$$[X_i^{+}(z), X_j^{-}(w)] =$$

$$(q - q^{-1})\delta_{ij}\{\delta(zw^{-1}q^{-c})k_{i+1}^{-}(w_{-})k_i^{-}(w_{-})^{-1} - \delta(zw^{-1}q^{c})k_{i+1}^{+}(z_{-})k_i^{+}(z_{-})^{-1}\}.$$

$$\{X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_j^{\pm}(w) - (q + q^{-1})X_i^{\pm}(z_1)X_j^{\pm}(w)X_i^{\pm}(z_2)+$$
$$X_j^{\pm}(w)X_1^{\pm}(z_1)X_i^{\pm}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0,$$
$$for \quad A_{ij} = -1.$$

Here $z_{\pm} = q^{\mp\frac{c}{2}}z$, A_{ij} are the entries of the Cartan matrix for $\mathfrak{sl}(n)$, and

(2.5)
$$\delta(z) = \sum_{n \in \mathbf{Z}} z^n.$$

Drinfeld's realization of the subalgebra $U_q(\widehat{\mathfrak{sl}(n)})$ is given by $x_i^{\pm}(z) = (q-q^{-1})^{-1}X_i^{\pm}(zq^i)$, $\psi_i(z) = k_{i+1}^{-}(zq^i)k_i^{-}(zq^i)^{-1}$ and $\varphi_i(z) = k_i^{+}(zq^i)k_{i+1}^{+}(zq^i)^{-1}$. [DF].

Drinfeld [D3] stated that the algebra $U_q(\widehat{\mathfrak{sl}(n)})$ is isomorphic to the one constructed by generators and relations with extended Cartan matrix of type $\hat{A}_{n-1}$.

The third and the fourth formulas above are defined on the completion of the

Let $R(z)$ be an element of $\text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ defined by

$$
R(z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j \\ i,j=1}}^{n} E_{ii} \otimes E_{jj} \frac{z-1}{q^{-1}z - q}
$$

(2.6)

$$
+ \sum_{\substack{i<j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji} \frac{z(q^{-1}-q)}{zq^{-1}-q} + \sum_{\substack{i>j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji} \frac{(q^{-1}-q)}{q^{-1}z-q}
$$

where q, z are formal variables. Then $R(z)$ satisfies the Yang-Baxter equation with a parameter:

(2.7) $$R_{12}(z)R_{13}(z/w)R_{23}(w) = R_{23}(w)R_{13}(z/w)R_{12}(z),$$

and R is unitary, namely

(2.8) $$R_{21}(z)^{-1} = R(z^{-1}).$$

The definition of $R(z)$ implies

$$
\lim_{z \to 0} R(z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j \\ i,j=1}}^{n} qE_{ii} \otimes E_{jj} + q^{-1}(q-q^{-1}) \sum_{\substack{i>j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji}
$$

$$
= q^{-1}R_{21},
$$

(2.9)

$$
\lim_{z \to \infty} R(z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j \\ i,j=1}}^{n} q^{-1}E_{ii} \otimes E_{jj} + q(q^{-1}-q) \sum_{\substack{i<j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji}
$$

$$
= qR^{-1},
$$

where R is the R-matrix of $U_q(\mathfrak{gl}(n))$ on $\mathbb{C}^n \otimes \mathbb{C}^n$[J3].

Faddeev, Reshetikhin and Takhtajan defined a Hopf algebra using this element $R(z)$. Reshetikhin and Semenov-Tian-Shansky obtained a central extension of an algebra, which is defined in the same way but with a R-matrix of a scalar multiple of $R(z)$ above. However these two algebras are isomorphic, which is implied in the proof of the main theorem in [DF].

We will denote by $U(\tilde{R})$ the algebra with central extension defined by the $R(z)$ above. The central extension is incorporated in shifts of the parameter z in $R(z)$.

Definition 2.2. $U(\tilde{R})$ *is an associative algebra with generators* $\{l_{ij}^{\pm}[\mp m], m \in \mathbf{Z}_+ \setminus 0 \quad and$

$l_{ij}^{+}[0], l_{ji}^{-}[0], 1 \leq j \leq i \leq n\}$. *Let* $l_{ij}^{\pm}(z) = \sum\limits_{m=0}^{\infty} l_{ij}^{\pm}[\pm m]z^{\pm m}$, *where* $l_{ij}^{+}[0] = l_{ji}^{-}[0] = 0$, *for* $1 \leq i << n$. *Let* $L^{\pm}(z) = (l_{ij}^{\pm}(z))_{i,j=1}^{n}$. *Then the defining relations are the following:*

$$
l_{ii}^{+}[0]l_{ii}^{-}[0] = l_{ii}[0]l_{ii}^{+}[0] = 1,
$$

$$
R(\frac{z}{w})L_1^{\pm}(z)L_2^{\pm}(w) = L_2^{\pm}(w)L_1^{\pm}(z)R(\frac{z}{w}),
$$

(2.10)

$$
R(\frac{z_+}{w})L_1^{+}(z)L_2^{-}(w) = L_2^{-}(w)L_1^{+}(z)R(\frac{z_-}{w}),
$$

where $z_\pm = zq^{\mp\frac{c}{2}}$. For the first formula of (2.10), we can expand $R(\frac{z}{w})$ in formal power series of either $\frac{z}{w}$ or $\frac{w}{z}$, but for the second formula of (2.10), it is expanded in the formal power series of $\frac{z}{w}$.

$U(\tilde{R})$ is a Hopf algebra: its coproduct is defined by

$$\Delta' L^\pm(z) = L^\pm(zq^{\mp(1\otimes\frac{c}{2})})\dot\otimes L^\pm(zq^{\pm(\frac{c}{2}\otimes 1)})$$

(2.11)

$$or \qquad \Delta'(l_{ij}^\pm(z)) = \sum_{k=1}^n l_{ik}^\pm(zq^{\mp(1\otimes\frac{c}{2})}) \otimes l_{kj}^\pm(zq^{\pm(\frac{c}{2}\otimes 1)}),$$

and its antipode is

(2.12)
$$S(L^\pm(z)) = L^\pm(z)^{-1}.$$

Note that the invertibilty of $L^\pm(z)$ follows from the properties that $l_{ii}^\pm$ are invertible and $L^+(0)$ and $L^-(0)$ are upper triangular and lower triangular operator-entried matrices respectively.

Theorem 2.1. $L^\pm(z)$ have the following unique decompositions:

(2.13)

$$L^\pm(z) = \begin{pmatrix} 1 & & & & 0 \\ e_{2,1}^\pm(z) & \ddots & & & \\ e_{3,1}^\pm(z) & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ e_{n,1}^\pm(z) & \cdots & e_{n,n-1}^\pm(z) & e_{n-1,n}^\pm(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm(z) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & k_n^\pm(z) \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & f_{1,2}^\pm(z) & f_{1,3}^\pm(z) & \cdots & f_{1,n}^\pm(z) \\ & \ddots & \ddots & \ddots & \vdots \\ & & & & f_{n-1,n}^\pm(z) \\ 0 & & & & 1 \end{pmatrix},$$

where $e_{i,j}^\pm(z)$, $f_{j,i}^\pm(z)$ and $k_i^\pm(z)(i > j)$ are elements in $U(\tilde{R})$ and $k_i^\pm(z)$ are invertible. Let

(2.14)
$$X_i^-(z) = f_{i,i+1}^+(z_+) - f_{i,i+1}^-(z_-),$$
$$X_i^+(z) = e_{i+1,i}^+(z_-) - e_{i+1,i}^-(z_+),$$

then $q^{\pm\frac{1}{2}c}, X_i^\pm(z), k_j^\pm(z), i = 1,\ldots,n-1, j = 1,\ldots,n$ satisfy the relations (2.3), (2.4) of $U_q(\widehat{\mathfrak{gl}(n)})$. The homomorphism

defined by (2.14) is an isomorphism.

Since $k_i^{\pm}(z)$ are invertible, the elements $e_{i,j}^{\pm}(z)$, $f_{j,i}^{\pm}(z)$ and $k_i^{\pm}(z)(i > j)$ are uniquely expressed in terms of the matrix coefficients of $L^{\pm}(z)$. In view of this analogy we call (2.13) Gauss decomposition of $L^{\pm}(z)$.

We note that as a corollary of Theorem 2.1 and Definition 2.1, one gets a realization of $U_q(\widehat{\mathfrak{sl}(n)})$ as a subalgebra of $U(\tilde{R})$.

Let $L(z) = L^+(zq^{-c})(L^-(z))^{-1}$, then

$$(2.16) \quad R(z/w)L_1(z)R(zq^{2c}/w)^{-1}L_2(w) = L_2(w)R(zq^{-2c}/w)L_1(z)R(z/w)^{-1}.$$

3. Spinor representation of $\hat{\mathfrak{g}}l(n)$ and Boson-Fermion Correspondence

For $\hat{\mathfrak{g}}l(n)$, we assume $n > 2$. This restriction is to avoid some nonessential but tedious complication caused by the selfdual property of the standard representation of $\mathfrak{s}l(2)$ on $\mathbb{C}^2$, which can be resolved.

Definition 3.1. *The Affine Clifford algebra is an associative algebra generated by $a_i(m)$ and $a^*(m)$, $(i = 1, .., n; m \in \mathbb{Z})$ with the commutation relations:*

$$a_i(m)a_j(l) + a_j(l)a_i(m) = 0,$$

$$a_i^*(m)a_j^*(l) + a_j^*(l)a_i^*(m) = 0,$$

$$(3.1) \quad a_i(m)a_j^*(l) + a_j^*(l)a_i(m) = \delta_{ij}\delta_{m,-l}1.$$

Definition 3.2. *The affine Heisenberg algebra is an associative algebra generated by $h_i(m)$, $m \neq 0, i = 1, ..., n$ and $m \in \mathbb{Z}$. The relations are*

$$h_i(m)h_j(l) - h_j(l)h_i(m) = 0, l \neq -m,$$

$$(3.2) \quad h_i(m)h_j(-m) - h_j(-m)h_i(m) = m\delta_{ij}, m \neq 0.$$

We now introduce the notation of formal power series. (For detail see [FLM].) For a vector space W, we denote that

$$(3.3) \quad W[[z, z^{-1}]] = \{\Sigma_{m \in \mathbb{Z}}v_m z^m \| v_m \in W\},$$

where z is a formal variable. For example, let W be also an algebra on $\mathbb{C}$, $\delta(z) = \Sigma_{m \in \mathbb{Z}}z^m$ as defined in (2.5) will be used frequently. Given two former power series $g_1(z) = \Sigma g_i(m)z^{-m}$ and $g_2(z) = g_2(m)z^{-m}$ over an algebra W, by the product of these two formal power series operators is well defined, we mean that the limit of each homogeneous component of $\Sigma_{m_1 < m < m_2}g_1(m)z^{-m}\Sigma_{l_1 < l < l_2}g_2(l)z^{-l}$, when l_1, m_1 go to negative infinity and m_2, l_2 go to positive infinity, exists. We also define the product as a formal power series with the corresponding limit as the coefficient of the corresponding homogeneous component. This complies with the definition of the composition of intertwiners in Section. For this paper, we define that the limit of the series of operators as the operator whose matrix coefficients are

For example, let $f(z)$ be a polymonial of z and z^{-1}, then $f(z)\delta(z)$ is well defined and

(3.4) $$f(z)\delta(z) = f(1)\delta(z).$$

Let E_{ij} be the standard basis of Lie algebra $\mathfrak{gl}(n)$. As showed in [G], $\hat{\mathfrak{gl}}(n)$ can be realized as $\mathfrak{gl}(n) \otimes [x, x^{-1}] \oplus C$, where C is the central element. Let $E_{ij}(z) = \Sigma E_{ij} \otimes x^n z^{-n}$ be the standard basis of the generating functions of $\hat{\mathfrak{gl}}(n)$, where z is a formal variable.

We define $a_i(z)$, $a^*(z)$ formal power series with coefficients in the corresponding algebra as

$$a_i(z) = \Sigma a_i(m)z^{-m},$$

(3.5) $$a_i^*(z) = \Sigma a_i^*(m)z^{-m}.$$

Let P be an n dimensional lattice $\Sigma \oplus \mathbb{Z}h(0)_i$, with the form that $(h(0)_i, h(0)_j) = \delta_{ij}$. Let $\mathbb{C}[\bar{P}]$ be the central extension of $\bar{P}$, the group algebra of P, such that $e^{h(0)_i}e^{h(0)_j} = (-1)^{(h(0)_i, h(0)_j)}e^{h(0)_j}e^{h(0)_j}$. Let $\bar{H}$ be an associative algebra generated by $h_i(m)$, $m \neq 0$ and $\mathbb{C}[\bar{P}]$, where $h_i(m)$ and $\mathbb{C}[\bar{P}]$ commute with each other. Let $h_i(z) = \Sigma h_i(m)z^{-m} + \partial_{h(0)_i}$, where $\partial_{h(0)_i}$ is the partial differential of $h(0)_i$.

Definition 3.3. *Spinor Fock space is the space of the subalgebra of affine Clifford algebra generated by $a_i(-m)$, $a^*(-m)$, $m > 0$ and $a_i^*(0)$.*

Definition 3.4. *Oscillator Fock space is the space of the subalgebra of $\bar{H}$ generated by $h_i(-n), n > 0$ and $\mathbb{C}[\bar{P}]$.*

In this paper, we define normal ordering : : as in [F1].

Proposition 3.1. [F] [FF] [F1] *Let $a_i(z), a^*(z)$ be as defined above and $E_{ij}(z)$ be the standard basis of $\hat{\mathfrak{gl}}(n)$. There is a representations of $\hat{\mathfrak{gl}}(n)$ given by the following on the spinor Fock space:*

(3.6) $$E_{ij}(z) \longrightarrow: a_i(z)a_j^*(z) : .$$

where : : denotes the normal ordering.

Proposition 3.2. [FK] [F1] *Let $h_i(z)$ as defined above, there is the Frenkel-Kac construction of a representation of $\hat{\mathfrak{gl}}(n)$ on the oscillator Fock space by vertex operators given by*

(3.7) $$E_{ii}(z) \longrightarrow h_i(z),$$

$$E_{ij}(z) \longrightarrow: exp(\bar{h}_i(z) - \bar{h}_j(z)) :, i \neq j,$$

where $\bar{h}_i(z) = \Sigma_{n \neq 0} 1/n h_i(-n)z^n + \partial_{h(0)_i} lnz + h(0)_i$. This representation is isomorphic to the representation of Proposition 3.1 above. The isomorphism is given as:

$$a_i(z) =: exp(\bar{h}_i(z)) :,$$

(3.8) $$a_i^*(z) =: exp(-\bar{h}_i(z)) : .$$

Here :: is defined in the standard way as in [F1].

The isomorphism can be easily proved. First, we shift the degree of $a_i(z)$ and $a_i^*(z)$ by $\pm 1/2$ respectively. Then by comparing the character, we get the proof[F1]. The propositions above give us the classical boson-fermion correspondence.

Proposition 3.3. V_{bf} *contains a unique copy of the highest weight representations of* $\hat{\mathfrak{gl}}(n)$ *with* C *acting as a multiplication of 1, the highest weight as any of the fundamental weights corresponding to any i-th fundamental weight of the classical part* $\mathfrak{sl}(n)$ *of* $\hat{\mathfrak{sl}}(n)$ *or zero weight and the action of* $\Sigma_{i=1,...,n}E_{ii}$ *as any integer* l *such that* $l = i(mod(n))$.

We know that $\hat{\mathfrak{gl}}(n) = \hat{\mathfrak{sl}}(n) \oplus \hat{\mathfrak{gl}}(1)$. For the above case, $\Sigma h_i(z) = \Sigma_{i=1,...n} :$ $a_i(z)a_i^*(z) :$ gives us $\hat{\mathfrak{gl}}(1)$, which commutes with $\hat{\mathfrak{sl}}(n)$.

Let e_i be the standard basis of $V = \mathbb{C}^n$ and e_i^* be the dual basis of the dual module V^*. Let $V(z)$ and $V^*(z)$ be the evaluation representation of $\hat{\mathfrak{gl}}(n)$, where z is a formal variable and the action of $E_{ij} \otimes t^n$ of $\hat{\mathfrak{gl}}(n)$ is given by the action of E_{ij} of $\mathfrak{gl}(n)$ with z^n scalar multiple.

Proposition 3.4. *Let*

$$\Phi^c = \Sigma a_i^*(z) \otimes e_i,$$

(3.9)
$$\Phi^{c*} = \Sigma a_i(z) \otimes e_i^*.$$

Then Φ^c *and* Φ^{c*} *are intertwiners:*

$$\Phi^c : V_{bf} \longrightarrow V_{bf} \otimes V_z,$$

(3.10)
$$\Phi^{c*} : V_{bf} \longrightarrow V_{bf} \otimes V_z^*.$$

This can be checked by direct calculation. This observation is the starting point of our approach to the spinor and oscillator representations. As in the finite dimensional cases [DF2], we explain the hidden structure behind all the constructions above.

Now we will derive the above construction without any specific realizations. In the next section, we will present a parallel q-deformation of this abstract construction.

We now start from an abstract module $V_{bf} = \Sigma_{l \in \mathbb{Z}} V^l$, where V^l is the highest weight representation with the $l(mod(n))$-th fundamental weight, with central extension 1 and the action of ΣE_{ii} as integer l. By the 0-th fundamental weight, we mean zero weight. V^l has infinite many copies of the highest weight representation of $\hat{\mathfrak{sl}}(n)$ corresponding to the the $l(mod(n))$-th fundamental weight of $\mathfrak{sl}(n)$ and with central extension 1. From the abstract representation theory of Kac-Moody algebras, we know the existance of those modules.

As we explain above, the representation of $\hat{\mathfrak{gl}}(n)$ can be derived on the the space $V_z = V[z, z^{-1}]$ with the action ΣE_{ii} as 1.

We refer this notation to [FLM], where it is introduced for vertex operator algebras. By $\bar{V}_z$, we mean the set of all vectors in the form of $\Sigma v_i f_i(z)$, where $v_i \in V$ form a basis of V and $f_i(z)$ are formal power series on z and z^{-1}.

Let V^* be the left dual module of V. Let F be an invariant vector in $V \otimes V^*$ of $\mathfrak{gl}(n)$, which is unique up to a scalar multiple. We normalize it, such that F is equal to the identity if it is identified as an element in $\text{End}(V)$.

Proposition 3.5. $F(z_1, z_2) = \{x | x = F f(z_1)\delta(z_1/z_2)\}$, *where* $f(z_1)$ *is a polynomial of* z_1 *and* z_1^{-1}, *is an invariant subspace of* $\bar{V}_{z_1} \otimes \bar{V}_{z_2}^*$.

Proposition 3.6. *Let $\bar{\Psi}^l_z$ and $\bar{\Psi}^{*l}_z$ be intertwiners as the following:*

(3.11)
$$\bar{\Psi}^l_z : V^{l+1} \longrightarrow V^l \otimes V_z,$$

$$\bar{\Psi}^{*l}_z : V^l \longrightarrow V^{l+1} \otimes V^*_z,$$

such that $\bar{\Psi}^l_z = \Sigma \bar{\Psi}^l_i(-m) \otimes v_i z^m$ and $\bar{\Psi}^{*l}_z = \Sigma \bar{\Psi}^{*l}_i(-m) \otimes v_i z^m$, where $\bar{\Psi}^l_i(m)$ and $\bar{\Psi}^{*l}_i(m)$ are operators shifting the degree by m. Then these operators are unique up to a scalar multiple.

However, from [FF], we know that not all the highest weight vectors are graded 0, but that most of them have negative grading. We will thus shift the grading of V^l, for $l < 0$ or $l > n$ by defining that the grad of the highest weight vector of V^l is equal to $-n((\Sigma_0^{|m|-1}) + j(m)$, where $l = mn + j$, $n - 1 > j > 0$. After the shifting, we will denote the new operators by Ψ^l_z and Ψ^{*l}_z.

We know that the correlation of those operators satisfying KZ equation[TK] [FR], the solutions for the KZ equation in this case can be obtained. On the other hand, because the the space of the solutions in this case is one dimensional, we can normalize them in such a way that all $< v_{V^{l-1}}, \Psi^l_{z_2}\Psi^{l+1}_{z_1}v_{V^l} >$(i=0,1...,n-1) are equal, all $< v_{V^{l+2}}, \Psi^{*l+1}_{z_2}\Psi^{*l}_{z_1}v_{V^l} >$(i=0,1...,n-1) are equal and all $< v_{V^l}, \Psi^l_{z_2}\Psi^{*l}_{z_1}v_{V^l} >$ (i=0,1,...,n-1) are equal to $1/(1 - z_1/z_2)F$, where v_{V^l} is the highest weight vector of V^l. More precisely, we can choose the normalization in the way that, their correlation functions are equal to the corresponding correlation functions of Φ^c and Φ^{c*}.

Let $\Psi_z = \Sigma_i \oplus \Psi^i_z$ and $\Psi^*_z = \Sigma_i \oplus \Psi^{*i}_z$, then Ψ_z and Ψ^*_z are intertwining operators from V_{bf} to $V_{bf} \otimes V_z$ and $V_{bf} \otimes V^*_z$ respectively. For any vector v in V, we can get $\Psi(v)(n)$ in $\text{End}(V_{bf})$ and the same of $\Psi^*(v^*)(n)$ for v^* in V^*.

Proposition 3.7.
$$(\Psi_{z_2} \otimes I)\Psi_{z_1} + P'(\Psi_{z_1} \otimes I)\Psi_{z_2} = 0,$$

(3.12)
$$(\Psi^*_{z_2} \otimes I)\Psi^*_{z_1} + P'(\Psi^*_{z_1} \otimes I)\Psi^*_{z_2} = 0,$$

$$(\Psi_{z_2} \otimes I)\Psi^*_{z_1} + P'(\Psi^*_{z_1} \otimes I)\Psi_{z_2} - \delta(z_1/z_2)F = 0,$$

where P' is the operator which maps $a_i z^m_1 \otimes b_j z^l_2$ to $b_j z^l_2 \otimes a_i z^m_1$.

Proof. We can first prove the relations on the level of the correlation of the highest weight vector. Because the left hand sides of the formulas above are intertwiners, we can prove the equality above for correlation functions of any two vectors. Thus we finish the proof.

Theorem 3.1. *The algebra generated by $\Psi(v)(m)$ and $\Psi^*(v^*)(m)$ is isomorphic to the Clifford algebra of Definition 3.2.*

The proof is the same as that for Proposition 3.2.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type A_n, B_n, C_n or D_n. Let e_i, f_i and h_i, $i = 1, ..., n$ be the basic generators of $\mathfrak{g}$ corresponding to the Cartan matrix. Let $(,)$ be the Killing form on $\mathfrak{g}$. Let $\mathfrak{r} = \Sigma h_i \otimes h_i + \Sigma_{\Delta_+} e_\alpha \otimes e_{-\alpha} + \Sigma_{\Delta_-} e_\alpha \otimes e_{-\alpha}$, where $\Delta_\pm$ are the sets of all the positive roots and the negative roots respectively, e_α is a root vector in $\mathfrak{g}_\alpha$ and $(h_i, h_j) = \delta_{ij}$, $(e_\alpha, e_{-\alpha}) = 1$. $\mathfrak{r}$ is the Casmir operator of $\mathfrak{g}$. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$. $\hat{\mathfrak{g}}$ has a concrete realization

Definition 3.5. *Let z be a formal variable. We define the elements $\hat{r}$ and $r(z)$ in $\hat{\mathfrak{g}} \otimes \hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}} \otimes \hat{\mathfrak{g}}[z, z^{-1}]$ as*

$$(3.13) \qquad \hat{r} = \Sigma_{i,m} h_i x^m \otimes h_i x^{-m} + \Sigma_{\alpha \in \Delta, i, m} e_\alpha x^m \otimes e_{-\alpha} x^{-m},$$

$$r(z) = \Sigma_{i,m} h_i x^m \otimes h_i x^{-m} z^{-m} + \Sigma_{\alpha \in \Delta, i, m} e_\alpha x^m \otimes e_{-\alpha} x^{-m} z^{-m}.$$

We see that $\hat{r}$ is basically like a Casimir operator.

Let $e_i, f_i, h_i (\ i = 0, 1, .., n)$ be the basic generators of $\hat{\mathfrak{g}}$ for the corresponding Cartan matrix of $\hat{\mathfrak{g}}$. Let M be any finite dimensional module of $\hat{\mathfrak{g}}$, Let $V_{\mu,k}$ be a highest weight module with the highest weight μ and central extension k of $\hat{\mathfrak{g}}$. k is a complex number. $V_{\mu,k}$ is a graded module such that e_0 and f_0 changes the degree by $+1$ and -1 respectively as we explained before.

Theorem 3.2. $\pi_{V_{\mu,k}} \otimes \pi_M(r)$ *maps* $V_{\mu,k} \otimes M$ *to* $V_{\mu,k} \hat{\otimes} M$, *commutes with* e_i, f_i, h_i, *for* $i \neq 0$, *and*

$$(3.14) \qquad [\pi_{V_{\mu,k}} \otimes \pi_M(e_0), \pi_{V_{\mu,k}} \otimes \pi_M(r)] = -2k(id) \otimes (\pi_M(e_0)),$$

$$[\pi_{V_{\mu,k}} \otimes \pi_M(f_0), \pi_{V_{\mu,k}} \otimes \pi_M(r)] = 2k(id) \otimes (\pi_M(f_0)).$$

By $V_{\mu,k} \hat{\otimes} M$, *we mean the set of vectors in the form of* $\Sigma_{n \leq 0} \mu(n) \otimes m_i$, *where* $\mu(n)$ *is a vector in* $V_{\mu,k}$ *of degree* n.

This follows from direct calculation, which is also a direct corollary of the corresponding assertion in the quantum case.

Let $V = \mathbb{C}^n$ be the fundamental representation of $\mathfrak{g}$ as the case for $\hat{\mathfrak{g}}l(n)$ explained above. This representation can be extended to a representation of $\hat{\mathfrak{g}}$. It is clear that the concrete realization of $\pi_{V_{\mu,k}} \otimes \pi_V(r(z))$ can give us explicitly the construction of the representation. That means, for a specific representation $V_{\mu,k}$, constructing a representation is equivalent to giving an explicit expression of $\pi_{V_{\mu,k}} \otimes \pi_V(r(z))$. This is the central idea to understand the classical spinor constructions. From now on in this section, we assume that $\mathfrak{g} = \mathfrak{g}l(n)$. Let $V = \mathbb{C}^n$ be a module of $\hat{\mathfrak{g}}l(n)$ as we defined in the section above. Let t be a real number such that $|t|$ is less than 1.

Theorem 3.3. *Let* $\mathfrak{F}$ *be the standard map* $V^* \otimes V$ *to* $\mathbb{C}$. $(\Psi(z) \otimes I)\Psi^*(zt) - 1/(1 - t)id \otimes F$ *and* $\lim_{t \to 1}(\Psi(z) \otimes I)\Psi^*(zt) - 1/(1 - t)id \otimes F$ *are well defined. As a map from* $V_{bf} \otimes V$ *to* $V_{bf} \otimes V[z, z^{-1}]$,

$$-\bar{r}(z) = (id \otimes I \otimes \mathfrak{F})(\lim_{t \to 1}(\Psi(z) \otimes I \otimes I)\Psi^*(zt) \otimes I - 1/(1 - t)id \otimes F \otimes I)$$

$$(3.15) \qquad = \pi_{V_{bf}} \otimes \pi_V(\hat{r}(z)).$$

As a map from $V_{bf} \otimes V$ *to* $V_{bf} \hat{\otimes} V$,

$$(3.16) \qquad -\bar{r} = \pi_{V_{bf}} \otimes \pi_V(D_z^{-1} \otimes 1)(\hat{r}(z)).$$

.

By the limit above, we mean that we would take the limit for each homogeneous component separately.

Proof. It is clear that the first assertion implies the second one. The limit we take above is equivalent to the normal ordering defined in [FF]. A direct calculation shows that $\bar{r}$ satisfies the property (3.14) of r on the tensor module. We can show by calculation that the images of the difference of the degree zero terms of the highest weight vectors is zero. However we know that the difference between $\bar{r}$ and

4. Quantum Spinor representation of $U_q(\mathfrak{gl}(\hat{n}))$ and Boson-Fermion Correspondence.

We assume $n > 2$ for the same reason explained in the previous section.

We will proceed to construct the spinor representation and explain the quantum boson-fermion correspondence for $U_q(\mathfrak{gl}(\hat{n}))$. The degeneration of such a abstract construction provides us the classical boson-fermion correspondence as in the section above.

Proposition 4.1. $U_q(\hat{\mathfrak{gl}}(n))$ *is isomorphic to an algebra generated by* $U_q(\hat{\mathfrak{sl}}(n))$, $g(n)$, $n \neq 0$ *and* K, *such that* K *is central,* $g(n)$ *commute with* $U_q(\hat{\mathfrak{sl}}(n))$ *and* $[g(l), g(m)] = \delta_{l,-m} mC$, *where* C *is the central element of* $U_q(\hat{\mathfrak{sl}}(n))$[DF].

Proof. From $U_q(\hat{\mathfrak{sl}}(n))$, we can obtain $\bar{L}^{\pm}(z)$, which satisfy (1.13a). There exist $g^{\pm}(z) = e^{\Sigma b(\pm) g(\mp m) z^{\pm m}} e^{\pm 1/2 K}, m > 0$, such that the operators $g^{\pm}(z) \bar{L}^{\pm}(z)$ stisfy the relation (2.10). This gives us an algebra homomorphism. The isomorphism follows from the same proof as in [DF].

With the proposition above and Lusztig's theorem about deformation of the category of highest weight representations[L], we have that the module V_{bf} can be deformed. There is such a construction of this module by Frenkel and Jing [FJ].

As in the last section, we will start from the abstract construction. We will denote this deformed module of V_{bf} by $V_{\mathfrak{B}F}$. $V_{\mathfrak{B}F}$ as a module of $U_q(\hat{\mathfrak{gl}}(n))$ can be decomposed into irreducible components: $V_{\mathfrak{B}F} = \Sigma_{l \in \mathbb{Z}} \oplus V^l$, where V^l is an irreducible component of $V_{\mathfrak{B}F}$, which is a deformation of the module V^l of $\hat{\mathfrak{gl}}(n)$ in section above. But V^l as a module of $U_q(\hat{\mathfrak{sl}}(n))$ has infinite copies of V_i, which is a highest weight representation of $U_q(\hat{\mathfrak{sl}}(n))$ with the i-th fundamental weight and central extension 1. Let $V_{\mathfrak{b}f} = \Sigma \oplus V_i$. Surely, we assume that V_i has the same grading shifting as in the classical case.

We now will present the evaluation representations of $U_q(\hat{\mathfrak{sl}}(n))$ on the fundamental representations of the subalgebra $U_q(\mathfrak{sl}(n))$ generated $E_i, F_i, K_i, i \neq 0$ from [DO].

We define a characteristic function $\theta^J(j)$ of a set J by $\theta^J(j) = 1 \, (j \in J), 0 \, (j \notin J)$. If J is omitted, it should be understood as $\mathbb{Z}_{\geq 0}$.

Fix a positive integer k such that $1 \leq k \leq n-1$. Let $I = \{i_1, \cdots, i_k\}$ be a subset of $\{0, 1, \cdots, n-1\}$. For $I = \{i_1, \cdots, i_k\}$ we put $s(I) = i_1 + \cdots + i_k$. Consider a vector space $V^{(k)}$ spanned by the vectors $\{v_I\}$. We introduce an algebra homomorphism $\pi^{(k)} : U_q(\hat{\mathfrak{sl}}(n)) \longrightarrow \text{End}(V^{(k)})$ by

$$\pi^{(k)}(E_i) v_I = v_{I \setminus \{i\} \cup \{i-1\}},$$

(4.1)
$$\pi^{(k)}(F_i) v_I = v_{I \setminus \{i-1\} \cup \{i\}},$$

$$\pi^{(k)}(K_i) v_I = q^{\theta^I(i-1) - \theta^I(i)} v_I,$$

for $i \neq 0$, and

$$\pi^{(k)}(E_0) v_I = v_{I \setminus \{0\} \cup \{n-1\}},$$

(4.2)

$$\pi^{(k)}(K_0)v_I = q^{\theta^I(n-1)-\theta^I(0)}v_I.$$

Here $v_{I\setminus\{i\}\cup\{i-1\}}$ should be understood as 0 if $i \notin I$ or $i-1 \in I$. As a module $V^{(k)}$ of $U_q(\mathfrak{sl}(n))$ is isomorphic to the irreducible highest weight module with highest weight corresponding to the k-th node of the Dynkin diagram of $\mathfrak{sl}(n)$.

Let $V_z^{(k)} = V^{(k)} \otimes \mathbb{C}[z, z^{-1}]$. We can lift $\pi^{(k)}$ to an algebra homomorphism $\pi_z^{(k)} : U_q(\tilde{\mathfrak{sl}}(n)) \longrightarrow \mathrm{E}nd(V_z^{(k)})$ as follows:

$$(4.3) \qquad \pi_z^{(k)}(E_i)(v_I \otimes z^n) = \pi^{(k)}(e_i)v_I \otimes z^{n+\delta_{i0}},$$

$$\pi_z^{(k)}(f_i)(v_I \otimes z^n) = \pi^{(k)}(f_i)v_I \otimes z^{n-\delta_{i0}},$$

$$\pi_z^{(k)}(t_i)(v_I \otimes z^n) = \pi^{(k)}(t_i)v_I \otimes z^n,$$

$$\pi_z^{(k)}(q^d)(v_I \otimes z^n) = v_I \otimes (qz)^n.$$

Then we have the following isomorphism of $U_q(\hat{\mathfrak{sl}}(n))$–modules:

$$C_\pm^{(k)} : \quad V_{z(-q)^{\mp n}}^{(k)} \xrightarrow{\sim} \left(V_z^{(n-k)}\right)^{*a^{\pm 1}}$$

$$(4.4) \qquad v_I \mapsto (-q)^{\pm s(I)}v_{I^c}^*.$$

Here I^c denotes the complement set of I in $\{0, 1, \cdots, n-1\}$ and $\{v_{I^c}^*\}$ signifies the dual basis of $\{v_{I^c}\} \subset V^{(n-k)}$. This gives us the evaluation representations we need. By $(V^{(k)})^{*a^{\pm 1}}$, we mean the right and the left dual of $V^{(k)}$ respectively.

Let Φ be an intertwiner:

$$\Phi : V_{\mathfrak{b}f} \longrightarrow V^1 \otimes V_{\mathfrak{b}f},$$

$$(4.5) \qquad \Phi(x) = e_1 \otimes \Phi_1(x) + ... + e_n \otimes \Phi_n(x),$$

where $x \in V_{\mathfrak{b}f}$ and $\{e_i\}$ is a basis for V.

Let Φ^* be the intertwiner:

$$\Phi^* : V_{\mathfrak{b}f} \longrightarrow V^{1*} \otimes V_{\mathfrak{b}f},$$

$$(4.6) \qquad \Phi^*(x) = \bar{e}_1^* \otimes \Phi_1^*(x) + ... + \bar{e}_n^* \otimes \Phi_n^*(x),$$

where V^{1*} is the left dual representation of V of $U_q(\hat{\mathfrak{sl}}(n))$, $x \in V_{\mathfrak{b}f}$ and $\{\bar{e}_i^*\}$ is a basis for V^{*1}.

Let $\bar{\Phi}^*$ be the intertwiner:

$$\Phi^* : V_{\mathfrak{b}f} \longrightarrow^* V^1 \otimes V_{\mathfrak{b}f},$$

(4.7)

where $^*V^1$ is the right dual representation of V of $U_q(\hat{sl}(n))$, $x \in V_{bf}$ and $\{e_i^*\}$ is a basis for $^*V^1$.

From now on, we will identify V with V^1.

There is an isomorphism from $^*V$ to V^* by $a \to q^{2\rho}a$, for $a \in^* V[\text{FR}]$, where ρ is the half sum of all the positive roots of the $U_q(sl(n))$ in $U_q(\hat{sl}(n))$ generated by E_i, F_i and $K_i (i \neq 0)$.

As in the previous case, we can identify $V^* \otimes V$ with $\text{End}(V)$. With this identification, we define an operator $\tilde{L} \in \text{End}(V) \otimes \text{End}(V_{bf})$:

$$(4.8) \qquad \tilde{\mathfrak{L}} = (\tilde{\mathfrak{L}}_{ij}) = ((D_q^2 \Phi_j)\bar{\Phi}_i^*).$$

$$(4.9) \qquad \tilde{\mathfrak{L}}\Delta(a) = ((D_q^2 \otimes 1)\Delta(a))\tilde{\mathfrak{L}}.$$

The shift comes from the shift of Φ_{q^2}.

Here we have a problem that whether the multiplication of two operators $D_{q^2}\Phi_i$ and $\bar{\Phi}_j^*$ is well defined, which we do not have in the finite dimensional case. We procede to deal with this problem. With the condition $|q| < 1$, this multiplication is well defined, when we use Corollary 4.2 below, which comes from the results of the correlation functions of those intertwiners [DO]. We assume, from now on, that $|q| < 1$.

Let $\Delta_j = j(n-j)/2n$, for $j = 0, ..., n-1$ and we extend this index cyclically by $\Delta_j = \Delta_{j+n}$.

Let $\bar{V}_j$, $j = 0, 1, 2, .., n-1$ equivalent to V_j as a module of $U_q(\hat{sl}(n))$. We extend the index cyclically by identifying $\bar{V}_j$ with $\bar{V}_{j+n}$. For an integer l, let $\bar{l}$ stand for the integer such that $l \equiv \bar{l} \mod n$, $0 \leq \bar{l} < n$. Set

$$I_{jk} = \{\overline{j-k}, \overline{j-k+1}, \cdots, \bar{j}\}.$$

Let $\Phi_{\bar{V}_j}^{V^{(k)}\bar{V}_{j-k}}(z)$ denotes an intertwiner from $\bar{V}_j$ to $V_z^{(k)} \otimes \bar{V}_{j-k}$. Then our normalization reads as follows ($0 \leq j < n$):

$$\Phi_{\bar{V}_j}^{V^{(k)}\bar{V}_{j-k}}(z)|v_j\rangle = z^{\Delta_{j-k}-\Delta_j} v_{I_{jk}\backslash\{j\}} \otimes |v_{j-k}\rangle + \cdots,$$

$$\Phi_{\bar{V}_j}^{(V^{(k)})^{*a^\pm}\bar{V}_{j+k}}(z)|v_j\rangle = z^{\Delta_{j+k}-\Delta_j} v_{(I_{j,n-k}\backslash\{j\})^c}^* \otimes |v_{j+k}\rangle + \cdots,$$

where v_i is the highest weight vector in $\bar{V}_i$.

Let

$$\Phi(z) = (1 \otimes D_z^{-1})\Phi = \Sigma \oplus \Phi_{\bar{V}_j}^{V^{(1)}\bar{V}_{j-1}}(z),$$

$$\Phi^*(z) = (1 \otimes D_z^{-1})\bar{\Phi}^* = \Sigma \oplus \Phi_{\bar{V}_j}^{(V^{(1)})^{*a^+}\bar{V}_{j+1}}(z),$$

$$(4.10) \qquad \bar{\Phi}^*(z) = (1 \otimes D_z^{-1})\Phi^* = \Sigma \oplus \Phi_{\bar{V}_j}^{(V^{(1)})^{*a^-}\bar{V}_{j+1}}(z)$$

We also have that

(4.11)
$$\bar{\Phi}^*(z) = 1 \otimes q^{-2\rho}\Phi^*(zq^{-2n}).$$

The matrix coefficients of the highest weight vector were obtained in [DO]. The normalization above is given in [DO].

We will present the commutation relations between those intertwiners. In order to do this, we need to use the correlation functions given by Date and Okada [DO].

Let
$$(z;p)_\infty = \prod_{j=0}^{\infty}(1 - zp^j),$$

From now on, we will always use $\langle,\rangle$ to denote the matrix coefficient of the corresponding highest weight vectors of the highest weight modules.

Proposition 4.2. [DO]

$$\langle\Phi_{\bar{V}_{j+k}}^{V_2^{(k)}\bar{V}_j}(z_2)\Phi_{\bar{V}_{j+k+k'}}^{V_1^{(k')}\bar{V}_{j+k}}(z_1)\rangle$$

(4.12)
$$= z_1^{\Delta_{j+k}-\Delta_{j+k+k'}}z_2^{\Delta_j-\Delta_{j+k}}\prod_{i=1}^{m}\frac{((-q)^{2i+|k-k'|-2}z_1/z_2;q^{2n})_\infty}{((-q)^{-2i-|k-k'|}z_1/z_2;q^{2n})_\infty}$$

$$\times \sum_{I\subset I_0^{(k+k')},|I|=k}(-q)^{s(I_0^{(k)})-s(I)}(z_1/z_2)^{\mu_j(I)}v_{(I_0^{(k+k')}\backslash I)[j]} \otimes v_{I[j]}.$$

$$m = \min(k, k', n - k, n - k'),$$

$$\mu_j(I) = \sharp\{i \in I|i + j \geq n\} - \sharp\{i \in I_0^{(k)}|i + j \geq n\},$$

$$I[j] = \{\overline{i_1 + j}, \cdots, \overline{i_k + j}\} \qquad for\ I = \{i_1, \cdots, i_k\}.$$

Here $I_0^{(k)} = \{0, 1, \cdots, k - 1\}$, if $k + k' > n$, we formally understand $I_0^{(k+k')} = \{0, \cdots, n-1\}\sqcup I_0^{(k+k'-n)}$, where the elements $0, \cdots, k+k'-n-1$ have multiplicities, and we assume $I_0^{(k+k'-n)} \subset I, I_0^{(k+k')} \backslash I$.

Let
$$P_i : V^{(k)} \otimes V^{(k')} \longrightarrow V^{(k)} \otimes V^{(k')}$$

be a linear map such that P_i keeps the copy of $V^{(i)}$ in $V^{(k)} \otimes V^{(k')}$ invariant and map other irreducible components of $U_q(\mathfrak{sl}(n))$ to zero.

We define matrices, on $V^{(k)} \otimes V^{(k')}$, a matrix $\bar{R}_{V^{(k)}V^{(k')}}(z)$ as

(4.14)
$$P\bar{R}_{V^{(k)}V^{(k')}}(z) = \sum_{i=\max(0,k+k'-n)}^{\min(k,k')}\rho_i(z)P_i,$$

where P denotes the permutation. The coefficients $\rho_i(z)$ satisfy

$$\frac{\rho_{i-1}(z)}{} = \frac{z - (-q)^{k+k'-2i+2}}{}$$

$\rho_{\min(k,k')}(z) = 1$ by our normalization of $\bar{R}_{V^{(k)}V^{(k')}}(z)$.

Let

$$\Theta_p(z) = (z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty.$$

We define

(4.15) $$R_{kk'}(z) = \rho^{(k,k')}(z)\bar{R}_{V^{(k)}V^{(k')}}(z),$$

$$\rho^{(k,k')}(z) = z^{-kk'/n+\min(k,k')}\frac{((-q)^b z^{-1};q^{2n})_\infty((-q)^s z;q^{2n})_\infty}{((-q)^b z;q^{2n})_\infty((-q)^s z^{-1};q^{2n})_\infty}\prod_{i=1}^m \frac{\Theta_{q^{2n}}((-q)^{2i+b}z^{-1})}{\Theta_{q^{2n}}((-q)^{2i+b}z)},$$

where $b = |k-k'|$, $s = \min(k+k', 2n-k-k')$, m is defined as $\min\{k, k', n-k, n-k\}$ and $k, k' = 1$ or $n-1$.

Let

(4.16) $$R^*_{kk'}(z) = (\mathrm{id}\otimes C_-^{(n-k')})R_{k,n-k'}(z(-q)^{-n})(\mathrm{id}\otimes C_-^{(n-k')})^{-1},$$

$$R^{**}_{kk'}(z) = (C_-^{(n-k)}\otimes C_-^{(n-k')})R_{n-k,n-k'}(z)(C_-^{(n-k)}\otimes C_-^{(n-k')})^{-1},$$

.

Corollary 4.2. *Let $k, k' = 1$ or $n-1$. Then*

$$\langle\Phi^{V_2^{(k)}\bar{V}_j}_{\bar{V}_{j+k}}(z_1)\Phi^{V_1^{(k')}\bar{V}_{j+k}}_{\bar{V}_{j+k+k'}}(z_2)\rangle = PR_{kk'}(z_1/z_2)\langle\Phi^{V_2^{(k')}\bar{V}_j}_{\bar{V}_{j+k'}}(z_2)\Phi^{V_1^{(k)}\bar{V}_{j+k'}}_{\bar{V}_{j+k+k'}}(z_1)\rangle,$$

$$\langle\Phi^{V_2^{(k)}\bar{V}_j}_{\bar{V}_{j+k}}(z_1)\Phi^{V_1^{(k')*a^{-1}}\bar{V}_{j+k}}_{\bar{V}_{j+k-k'}}(z_2)\rangle$$

$$= PR^*_{kk'}(z_1/z_2)\langle\Phi^{V_2^{(k')*a^{-1}}\bar{V}_j}_{\bar{V}_{j-k'}}(z_2)\Phi^{V_1^{(k)}\bar{V}_{j-k'}}_{\bar{V}_{j+k-k'}}(z_1)\rangle,$$

(4.17) $$\langle\Phi^{V_2^{(k)*a^{-1}}\bar{V}_j}_{\bar{V}_{j-k}}(z_1)\Phi^{V_1^{(k')*a^{-1}}\bar{V}_{j-k}}_{\bar{V}_{j-k-k'}}(z_2)\rangle$$

$$= PR^{**}_{kk'}(z_1/z_2)\langle\Phi^{V_2^{(k')*a^{-1}}\bar{V}_j}_{\bar{V}_{j-k'}}(z_2)\Phi^{V_1^{(k)*a^{-1}}\bar{V}_{j-k'}}_{\bar{V}_{j-k-k'}}(z_1)\rangle.$$

In the neighborhood of $|z_1/z_2| = 1$, both sides of the second formula above with $k = k'$ have a simple pole at $z_1 = z_2$. Its residue is given by

$$PRes_{z_1=z_2}\langle\Phi^{V_2^{(k)}\bar{V}_j}_{\bar{V}_{j+k}}(z_1)\Phi^{V_1^{(k)*a^{-1}}\bar{V}_{j+k}}_{\bar{V}_j}(z_2)\,d\left(\frac{z_1}{z_2}\right)\rangle$$

(4.18) $$= h^{(k)}\sum_{I\subset I_0^{(n)},|I|=k} v_I \otimes v_I^*,$$

where

$$h^{(k)} = \frac{(q^{2n-2};q^{2n})_\infty}{(q^{2n};q^{2n})_\infty}\prod_{i=1}^{\min(k,n-k)-1}\frac{(q^{2n-2i-2};q^{2n})_\infty}{(q^{2i};q^{2n})_\infty}.$$

Let

$$f(z_1/z_2) = \frac{(z_2/z_1;q^{2n})_\infty}{(q^{-2}z_2/z_1;q^{2n})_\infty},$$

$$F(z_1/z_2) = \frac{((q)^{2n-2}z_2/z_1;q^{2n})_\infty}{(q^{2n}z_2/z_1;q^{2n})_\infty}.$$

Note that $(1 - z_2/z_1 q^{-2})f(z_1/z_2) = (1 - z_2/z_1)/F(z_1/z_2)$

Let $R_{1,1}(z_1/z_2)$, $R^{**}_{n-1,n-1}(z_1/z_2)$ and $R^*_{1,n-1}(z_1/z_2)$ be matrices as defined above

Corollary 4.2. [FR][DFJMN] [DO] *Let z, z_1 and z_2 be formal variables. $\Phi(z)$ and $\bar{\Phi}^*(z)$ satisfy the commutation relations:*

$$1/f(z_1/z_2)\Sigma\Phi_j(z_1)\Phi_i(z_2)e_i \otimes e_j =$$

$$P'1/f(z_1/z_2)(R_{1,1}(z_1/z_2))(\Sigma\Phi_{j'}(z_2)\Phi_{i'}(z_1)e_{i'} \otimes e_{j'}),$$

$$1/f(z_1/z_2)\Sigma\bar{\Phi}_i^*(z_1)\bar{\Phi}_j^*(z_2)e_j^* \otimes e_i^* =$$

(4.19)
$$P'1/f(z_1/z_2)(R_{n-1,n-1}^{**}(z_1/z_2)(\Sigma\bar{\Phi}_i^*(z_2)\bar{\Phi}_j^*(z_1)e_j^* \otimes e_i^*),$$

$$1/F(z_1/z_2)\Sigma\bar{\Phi}_j(z_1)\Phi_i^*(z_2)e_i^* \otimes e_j - 1/(1 - z_1/z_2)F =$$

$$P'1/F(z_1/z_2)(R_{1,n-1}^*(z_1/z_2))(\Sigma\Phi_j^*(z_1)\bar{\Phi}_i(z_2)e_i \otimes e_j^*) - (z_1/z_2)/(z_1/z_2 - 1)F,$$

$$\lim_{z_1 \to 1} \lim_{z_1 \to z_2, |z_1| < |z_2|} (z_1 - z_2)\Sigma\Phi_i(z_1)\bar{\Phi}_i^*(z_2)e_i \otimes e_i^* = P\frac{(q^{2n-2};q^{2n})_\infty}{(q^{2n};q^{2n})_\infty}F,$$

where $f(z_1/z_2)$ and $F(z_1/z_2)$ are expanded in the power series of z_2/z_1 on the left hand side of the the formulas above but in the power series of z_1/z_2 on the right hand side. $F = \Sigma e_i \otimes e_i^*$.

Proof. Our arguement is based on the formlas in [DO] of the correlation functions.

The argument for the first two formulas is straightforward. First, we know that the formula is true on the level of correlation functions of the highest weight vectors of $V_{\mathfrak{b}f}$, due to the fact that after we factor out those functions as above, the correlation functions of the operators on both sides of highest weight vectors are polymonials as showed in [DO]. On the other hand, both sides are intertwiners, this relation thus can be proved to be true for the matrix coefficient of any two vectors, thus both sides are equal.

As for the formula for the commutation relations between Φ_i and $\bar{\Phi}_j^*$, the argument goes as follows: the first part is the same as that of above, namely, the formula is valid on the level of correlation functions of the highest weight vectors; secondly, $\Sigma_{n \in \mathbb{Z}}(z_1/z_2)^n F$ is an invariant vector, thus $\mathrm{id} \otimes \Sigma_{n \in \mathbb{Z}}(z_1/z_2)^n F$ is also an intertwiner. Thus the difference of two sides is also an intertwiner, then we can show that the third formula is valid on the level of correlation functions of any two vectors of the two sides of the third formula. Thus it is valid.

Basically the idea appeared in [FR] [DFJMN], which is to study the relations between the correlation functions.

The commutation relation of each homogeneous component of $\Phi_i(z)$, $\bar{\Phi}_j^*(z)$ described by using R-matrix will degenerate into the commutation relation with $\delta(z)$.

The locations of the poles of the correlation functions of $\Phi_j(z_1)\bar{\Phi}_i^*(z_2)$ do not include the line $z_1q^2 = z_2$. From the commutation relations and the condition that $|q| < 1$, the multiplication of $\bar{\Phi}^*(z)$ and $\Phi(zq^2)$ is well defined. Thus $(D_{q^2}\Phi_i)\bar{\Phi}_j^*e_j^* \otimes e_i = (1 \otimes 1 \otimes D_z^{-1})(1 \otimes \Phi(zq^2))\bar{\Phi}^*(z)$ is well defined.

We know that both $V \otimes V_i$ and $V_{q^2} \otimes V_i$ are irreducible [FR] [KKMMNN] [JM]. This shows that the dimension of the space of the operators $X : V \otimes V_{\mathfrak{b}f} \longrightarrow V_{q^2} \otimes V_{\mathfrak{b}f}$, which satisfy the following relation :

is n. Thus the difference between $\tilde{L}$ and L is a constant factor if we restrict it to each irreducible component of $V \otimes V_i$, which can be determined by looking at their actions on the highest weight vectors. By looking at the homogeneous component of degree 0 of the correlation functions, we know that there is a universal factor c for all the irreducible components, which can be derived by comparing the actions of $\mathfrak{L}$ and $\tilde{L}$ on the highest weight vectors.

Let $\tilde{\mathfrak{L}}(z) = (1 \otimes \Phi(zq^2))\bar{\Phi}^*(z)$.

Theorem 4.1.

(4.20)
$$\mathfrak{L}(z) = c(D_z \otimes 1)\tilde{\mathfrak{L}},$$

where $c = \frac{tr(v_0, (D_{q^2}\bar{\Phi}^*)\Phi v_0)}{tr(v_0, \mathfrak{L}(1)v_0)}$ and v_0 is the highest weight vector of V_0.

We can remove the c in (4.11) by renormalizations of Φ and $\bar{\Phi}^*$.

From the definition of $\mathfrak{L}(z)$, we know that $\tilde{\mathfrak{L}}(z)$ can be decomposed into the product of $\mathfrak{L}^+(z)$ and $\mathfrak{L}^-(z)^{-1}$, which reminds us the polar decomposition. With the fermionic realizations, we could factor out $\mathfrak{L}^+(z)$ easily by looking at $\tilde{\mathfrak{L}}(z)v_i$, where v_i are the highest weight vectors for $V_{\mathfrak{b}f}$. This decomposition is unique.

Now, let's consider $U_q(\hat{\mathfrak{gl}}(n))$. As in [DF], by adding an extra Heisenberg algebra $h(n)$, which commutes with $U_q(\hat{\mathfrak{sl}}(n))$, we would obtain the representation of $U_q(\hat{\mathfrak{gl}}(n))$. From [DF] and (1.11b)

Proposition 4.2. *There exists complex numbers $a(n)$ such that*

$$\bar{\mathfrak{L}}^+(z) = (\mathfrak{L}^+(z) \otimes e^{\sum_{m \in \mathbb{Z}_+} a(m)h(-m)(z)^m})^{-1},$$

(4.21)
$$\bar{\mathfrak{L}}^-(z) = (\mathfrak{L}^-(z) \otimes e^{\sum_{m \in \mathbb{Z}_+} a(-m)h(m)z^{-m}})^{-1},$$

satisfy the commutation relation (2.10).

These operators acting on the tensor of the Fock space of $H(-m)$ and $V \otimes V_{\mathfrak{b}f}$. We will denote it by V_{gl}. $V_{gl} = V_{\mathfrak{b}f} \otimes V_h$, where V_h is the module generated by the extra Heisenberg algebra of $U_q(\hat{\mathfrak{gl}}(n))$. Let's give the same Gauss decomposition to these new operators. Then the action of the product of the zero component of the diagonal components of their decomposition, which we denote by T and T^{-1}, are 1. Let A be a group algebra generated by a lattice $\mathbb{Z}a$. Let $\bar{V} = \Sigma \oplus V_{gl}^i \otimes e^{mn+i}$, $m \in \mathbb{Z}$ in the space $V_{gl}^i \otimes A$, where $V_{gl}^i = V_i \otimes V_h$. We define $\bar{V}$ to be a module of $U_q(\hat{\mathfrak{gl}}(n))$, such that all other elements acting only on V_{gl}, but the action of T on $V_{gl} \otimes e^{ma}$ is a multiplication of q^m.

Proposition 4.4. $T^{\pm 1/2}\bar{\mathfrak{L}}^{\pm}(z)$ *gives us representation $L^{\pm}(z)$ of $U_q(\hat{\mathfrak{gl}}(n))$ on $\bar{V}$. $\bar{V}$ is equivalent to $V_{\mathfrak{B}F}$.*

Let $\bar{h}(z) = \Sigma a(-m)h(m)z^{-m}/(q^2-1)$ and $\bar{h}(z_1)\bar{h}(z_2) = g(z_1/z_2) + : \bar{h}(z_1)\bar{h}(z_2) :$

Definition 4.1. *Let z, z_1 and z_2 be formal variables. Affine Quantum Clifford algebra is defined as an associative algebra generated by $\psi(z) = (\psi_i(z)) = (\Sigma \psi_i(m) z^{-m})$ and $\psi^*(z) = (\psi_i^*(z)) = (\Sigma \psi_i^*(m) z^{-m})$, $0 < i < n+1$ satisfying the commutation relations:*

$$1/(f(z_1/z_2)/G(z_1/z_2)) \Sigma \psi_j(z_1) \psi_i(z_2) e_i \otimes e_j =$$
$$P' 1/(f(z_1/z_2)/G(z_2/z_1))(R_{1,1}(z_1/z_2))(\Sigma \psi_{i'}(z_2) \psi_{j'}(z_1) e_{j'} \otimes e_{i'}),$$
$$1/(f(z_1/z_2)/G(z_1/z_2)) \Sigma \psi_j^*(z_1) \psi_i^*(z_2) e_i^* \otimes e_j^* =$$

(4.22)
$$P' 1/(f(z_1/z_2)/G(z_2/z_1))(R_{n-1,n-1}^{**}(z_1/z_2)(\Sigma \psi_i^*(z_2) \psi_j^*(z_1) e_j^* \otimes e_i^*),$$
$$1/(F(z_1/z_2)G(z_1/z_2)) \Sigma \psi_j(z_1) \psi_i^*(z_2) e_i^* \otimes \bar{e}_j - 1/(1 - z_2/z_1) F =$$
$$P' 1/(F(z_1/z_2)G(z_2/z_1))(R_{1,n-1}^*(z_1/z_2))(\Sigma \psi_j^*(z_1) \psi_i(z_1) \bar{e}_i \otimes e_j^*)$$
$$-(z_1/z_2)/(z_1/z_2 - 1) F,$$

$$\lim_{z_1 \to 1} \lim_{z_1 \to z_2, |z_1| < |z_2|} 1/(z_1 - z_2) \Sigma \psi_i(z_1) \psi_i^*(z_2) e_i^* \otimes e_i = \frac{(q^{2n-2}; q^{2n})_\infty}{(q^{2n}; q^{2n})_\infty} F/G(1),$$

where the functions of the left and the right hand sides are expanded in z_2/z_1 and z_1/z_2 respectively.

Theorem 4.2. *Quantum Clifford algebra is isomorphic to the algebra generated by $\Phi(z) \otimes e^{-\bar{h}(z)} \otimes e^{-a}$ and $\Phi(z)^* \otimes e^{\bar{h}(z)} \otimes e^a$ on $\bar{V}$.*

Proof. It is straight forward to show that the map from $\psi(z)$ to $\Phi(z) \otimes e^{-\bar{h}(z)} \otimes e^{-a}$ and $\psi^*(z)$ to $\Phi(z)^* \otimes e^{\bar{h}(z)} \otimes e^a$ is a surjective algebra homomorphism. Because $\rho^{kk}(z)$ for $k = 1$ and $k = n - 1$ has a factor in the form of $z(1 - z^{-1})/(1 - z) = -1$ and (2.9), if we shift the degree of $\psi(z)$ and $\psi^*(z)$ by $\pm 1/2$ as in the classical case, we can define the Fock space as in Definition 3.3, then derive the character with the calculation based on the R-matix on $\mathbb{C}^n \otimes \mathbb{C}^n$ given in Section 2 on the specific basis we chose. By comparing the character, we can prove the isomorphism.

In definition 4.1, we expect that the complicated functions showing in Definition 4.1 will conceal each other, such that we would get manageable and easy formulas. If this hypothesis is true, it hints that, as in the classical case, we should look at the corresponding case of $\mathfrak{g}l(n)$ instead of $\mathfrak{s}l(n)$, which should make the case much simpler.

With Theorem 4.2, we actually can start from the abstract algebra defined in Definition 4.1. Then we can derive L, which leads to the realization of $L^\pm(z)$ as defined in Section 2. From Theorem 2.1 in Section 2, through the Gauss decomposition of $L^\pm(z)$, we obtain all the quantum bosons out of $L^\pm(z)$. Thus we obtain the realization of the quantum boson-fermion correspondence in one direction. But we can not write explicit simple formulas due to the difficulties coming from the polar decomposition of $L(z)$ and the Gauss decomposition of $L^\pm(z)$, however the case for $U_q(\mathfrak{s}\hat{l}(2))$ can be solved in a relatively easy way through computation.

On the other hand, based on the work Koyayma [K] and Frenkel-Jing construction, we can write down partially the realization the quantum fermions in Bosons. With our results, we expect that a complete formula is very possible, if we consider $\mathfrak{g}l(n)$ instead of $\mathfrak{s}l(n)$ as we explain in Remark 1.

Let $\bar{F}(z_1/z_2) = (1 - z_2/z_1 q^{-2}) F(z_1/z_2)$ Let $H(z) = \Sigma H(n) z^{-n}$, $n \neq 0$ be an Heisenberg algebra such that : $e^{H(z_1)} :: e^{-H(z_2)} := 1/(\bar{F}(z_1/z_2)) : e^{H(z_1) - H(z_2)} :$. Let $\bar{\Psi}(z) = \Phi(z) \otimes : e^{-H(z)} :$, $\bar{\Psi}^*(z) = \Phi^*(z) \otimes : e^{H(z)} :$. Let $\tilde{V}$ be the pace of tensor

Theorem 4.3. *Let z, z_1 and z_2 be formal variables. $\bar{\Psi}(z)$ and $\bar{\Psi}^*(z)$ satisfies the following relations:*

$$\Sigma\bar{\Psi}_j(z_1)\bar{\Psi}_i(z_2)e_i \otimes e_j =$$

$$(1 - z_1/z_2)P'\bar{F}(z_2/z_1)/(f(z_1/z_2))(R_{1,1}(z_1/z_2))(\Sigma\bar{\Psi}_{i'}(z_2)\bar{\Psi}_{j'}(z_1)e_{j'} \otimes e_{i'}),$$

$$\Sigma\bar{\Psi}_j^*(z_1)\bar{\Psi}_i^*(z_2)\bar{e}_i^* \otimes \bar{e}_j^* =$$

(4.23) $(1-z_1/z_2)P'\bar{F}(z_2/z_1)/(f(z_1/z_2))(R_{n-1,n-1}^{**}(z_1/z_2)(\Sigma\bar{\Psi}_i^*(z_2)\bar{\Psi}_j^*(z_1)\bar{e}_j^*\otimes\bar{e}_i^*),$

$$\Sigma\bar{\Psi}_j(z_1)\bar{\Psi}_i^*(z_2)e_i^* \otimes \bar{e}_j - 1/(1 - z_2/z_1)F(1 - q^{-2}) =$$

$$P'1/(F(z_1/z_2)\bar{F}(z_2/z_1)))(R_{1,n-1}^*(z_1/z_2))(\Sigma\bar{\Psi}_i^*(z_2)\bar{\Psi}_j(z_1)\bar{e}_j \otimes e_i^*)$$

$$-(z_1/z_2)/(z_1/z_2 - 1)(1 - q^{-2})F,$$

$$\lim_{z_1 \to 1}\lim_{z_1 \to z_2,|z_1|<|z_2|} 1/(z_1 - z_2)\Sigma\bar{\Psi}_i(z_1)\bar{\Psi}_i^*(z_2)e_i^* \otimes \bar{e}_i =$$

$$(1 - q^{-2})F,$$

where the functions of the left and the right hand sides are expanded in z_2/z_1 and z_1/z_2 respectively.

The proof comes from straight calculation.

Discussion.

Theorem 4.3 gives a realization of an algebra, which has the same definiton formulas for form factors in quantum field theories[Sm], but the R-matrix here is different with a function factor.

To construct local operators in the theory of fromal factors is a very important problem[Sm]. If we consider the case of $U_q(\hat{sl}(n))$, and if we consider the intertwiners in (1.18) and (1.19), which we will call right intertwiners, as the basic generators to define form factors, it is known that certain compositon of the inetrtwiners of type as in (1.14) and (1.15), which we will call left intertwiners, gives local operators. However, how to derive the left intertwiners from the right intertwiners is a problem. With the spinor constrution we derive above, it is clear that we can derive the operator $\mathfrak{L}^{\pm}(z)$ through Gauss decomposition of $\mathfrak{L}(z)$ constructed out of the right intertwiners. The proper composition of $\mathfrak{L}^{\pm}(z)$ with the rihgt intertwiners obviously gives us the left intertwiners. Thus our construction provides a way to obtain local operators directly from the algebra, which defines form factors. Moreover, we propablely can modify the algebra with an extra Heisenberg algebra similar to the case in Theorem 4.3, such that we can derive in the same way operators $l^{\pm}(z)$ comimg from the Gauss decomposition of the operator $l(z)$ built out of those modified right intertwiners to derive operators similar to the left intertwiners, which, however, simply commutes with those modified right intertwiners, which is basically the definiton property of local operators.

As a natural continuation, we are expecting to apply the same idea to under-covering the underlying structure of the so-called vertex operator algebras, which should lead us to the corresponding deformed structure, an axiomatic formulation quantum vertex operator algebra via the representation theory and the structure

should provide a proper mathematical setting to understand the massive quantum field theory in theoretical physics.

Recently, there appeared two papers by E. Stern [St] and M. Kashiwara, T. Miwa, and E. Stern(q-alg/9508006), where they study in detail basically the same Fock space as the Fock space defined by the quantum Clifford algebra in this paper.

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